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Zero dimensionality and monotone normality

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Abstract

A proof is given that every separable, compact, monotonically normal space X is the continuous image of a zero dimensional one which shows that X is the continuous image of a linearly ordered one. © 1998 Elsevier Science B.V.

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Nikiel [1] has conjectured that every compact monotonically normal space X is the continuous image of a compact linearly ordered space.

In [2] I prove this conjecture if X is also separable and zero dimensional. Here I prove the conjecture for separable X by proving:

Theorem 2. *Suppose X is a separable, compact, monotonically normal space. Then there is a separable, compact, zero dimensional, monotonically normal space Δ and a continuous map π from Δ onto X.*

The rest of the paper consists of a proof of this theorem. It mimics the proof given in [2] without assuming zero dimensionality. My hope is that generalizing the proof in this way may help someone to prove Nikiel's conjecture.

In Section 1 a *breakdown* is defined which partitions X into what we might call *atoms* and I then prove (Theorem 1) that there is a breakdown of X whose atoms each have cardinality at most two. In Section 2, after some preliminary work, constructions of types 1 and 2 are defined. These are used in Section 3 to define a Δ and π with the desired properties.

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1. Breakdown with atoms

Assume X is separable, compact, and monotonically normal.

The latter means [3] that points are closed and that X has an "MN operator". That is, for every closed A in X and open U with $A \subset U$, there is an open $H(A,U)$ with $A \subset H(A,U) \subset U$ such that:

(1) (normality) $A \cap V = \emptyset$ and $B \cap U = \emptyset$ imply $H(A,U) \cap H(B,V) = \emptyset$.

(2) (monotoniticy) $A \subset B$ and $U \subset V$ imply $H(A,U) \subset H(B,V)$.

If $A = \{x\}$ we simplify to just $H(x,U)$.

Let S be a countable dense subset of X.

If $\mathcal{G}$ is a finite open cover of X and $x \in X$, let

$$\mathcal{G}(x) = \{G \mid x \in G \text{ and either } G \in \mathcal{G} \text{ or } G = X - \overline{G'} \text{ for } G' \in \mathcal{G}\}.$$

Let $\mathcal{G}^*$ be a finite open cover of X such that for all $G \in \mathcal{G}^*$ there is $x \in X$ such that $G = H(x, \bigcap \mathcal{G}(x))$.

Call $\mathcal{F} = \bigcup\{\mathcal{F}_n \mid n \in \omega\}$ a *breakdown* of X provided:

(1) For $n \in \omega$, $\mathcal{F}_n$ is a finite cover of X by nonempty open sets.

(2) For $n \in \omega$, $\mathcal{F}_{n+1}$ is a refinement of $(\bigcup_{m \leqslant n} \mathcal{F}_m)^*$.

(3) If $p \neq q$ in S, there are F and F' in $\mathcal{F}$ having disjoint closures with $p \in F$ and $q \in F'$. Also $p \in S$ and $\{p\}$ is open imply $\{p\} \in \mathcal{F}$.

Let

$$\mathcal{K} = \left\{ \bigcap_{n \in \omega} F_n \mid F_n \in \mathcal{F}_n \text{ and } \overline{F_{n+1}} \subset F_n \right\}.$$

By (2), for all $x \in X$ there is $K \in \mathcal{K}$ with $x \in K$ and the members of $\mathcal{K}$ are disjoint. By (3), if $K \neq \{x\}$ for some $x \in S$, then x is a limit point of $S - K$.

Let

$$\mathcal{K}_3 = \{K \in \mathcal{K} \mid |K| \geqslant 3\}.$$

If $K \in \mathcal{K}_3$ choose distinct K_0, K_1, K_2 from K and disjoint open sets $W_0(K)$, $W_1(K)$, $W_2(K)$ with $K_i \in W_i(K)$ for all $i < 3$. For $i < 3$ let $V_i(K) = H(K_i, W_k(K))$ and $V_i'(K) = H(K_i, V_i(K))$. Since each K_i is a limit point of $S - K$ we can choose $F(K)$ and $F'(K)$ from $\mathcal{F}$ such that $K \subset F(K) \subset \overline{F(K)} \subset F'(K)$ and, for all $i < 3$, $H(K_i, V_i'(K)) - F'(K) \neq \emptyset$.

Lemma 1. *If $K \neq K'$ in $\mathcal{K}_3$, there is at most one $i < 3$ such that $V_i(K') \cap K \neq \emptyset$.*

Proof. There are $m \in \omega$ and disjoint F and F' from $\mathcal{F}_m$ with $K \subset F$ and $K' \subset F'$. There are $n > m$ and $F_n \in \mathcal{F}_n$ with $K \subset F_n \subset \overline{F_n} \subset F$. Thus there are $x \in F$ and $F_{n+1} \in \mathcal{F}_{n+1}$ with $K \subset F_{n+1} \subset H(x,F)$. Since $x \in W_i(K')$ for at most one $i < 3$, $H(K_j', W_j(K')) \cap H(x,F) = \emptyset$ for all $j \neq i$ in 3. $\square$

Lemma 2. *There do not exist infinitely many $K \in \mathcal{K}_3$ with the same $F(K)$ and $F'(K)$. (So $\mathcal{K}_3$ is countable.)*

Proof. Suppose $\{K(t) \mid t \in \omega\} \subset \mathcal{K}_3$ and there are F and F' with $F = F(K(t))$ and $F' = F'(K(t))$ for all $t \in \omega$.

If $i < 3$ and $j < 3$ define $I_{ij} = \{\{s,t\} \subset \omega \mid s \neq t$ and $s < t$ implies $K(t) \cap V_k(K(s)) = \emptyset$ if $k \neq i$ while $K(s) \cap V_k(K(t)) = \emptyset$ if $k \neq j\}$. By Ramsey's theorem [4], there are an infinite $T \subset \omega$, an $i < 3$, and a $j < 3$ such that $\{s,t\} \in I_{ij}$ for all $s \neq t$ in T. There is $k < 3$ with $k \neq i$ and $k \neq j$. Thus, for all $s \neq t$ in T, $K(t) \cap V_k(K(s)) = \emptyset$ and $\{V_k'(K(t) \mid t \in T\}$ are disjoint. For all $t \in T$ choose $p_t \in (H(K(t)_k, V_k'(K(t)) - F')$. Let p be a limit point of $\{p_t \mid t \in T\}$; $p \notin F'$ and $p \in V_k'(K(t))$ for at most one $t \in T$. Therefore, since H is an MN operator, $H(p, X - \overline{F}) \cap H(K(t)_k, V_k'(K(t)) = \emptyset$ for all but perhaps one t contradicting p is a limit point of $\{p_t \mid t \in T\}$.

Theorem 1. *There is a breakdown of X for which $\mathcal{K}_3 = \emptyset$.*

Proof. Otherwise, for each countable ordinal α, by induction we select a breakdown $\mathcal{F}(\alpha) = \bigcup\{\mathcal{F}_n(\alpha) \mid n \in \omega\}$ satisfying (1)–(3) among other things as follows. Then define $\mathcal{K}(\alpha)$ and $\mathcal{K}_3(\alpha)$ for this $\mathcal{F}(\alpha)$ exactly as $\mathcal{K}$ and $\mathcal{K}_3$ were defined for $\mathcal{F}$. For $K \in \mathcal{K}_3(\alpha)$ for some $\alpha < \omega_1$, choose K_i, $W_i(K)$, $V_i(K)$ and $V_i'(K)$ as before for each $i < 3$. By our construction which follows, $K \subset X$ can belong to $\mathcal{K}_3(\alpha)$ for at most one α; choose $F(K)$ and $F'(K)$ with reference to this α.

Let $\mathcal{F}(0)$ be some arbitrary breakdown of X.

Suppose $\mathcal{F}(\beta)$ has been chosen and $\alpha = \beta + 1$. If $K \in \mathcal{K}_3(\beta)$, let

$$\mathcal{G}(K) = \{W_i(K) \mid i < 3\} \cup \left\{X - \bigcup_{i<3} \overline{V_i(K)}\right\};$$

$\mathcal{G}(K)$ is a finite open cover of X. Since by Lemma 2, $\mathcal{K}_3(\beta)$ is countable, we can index $\{\mathcal{G}(K) \mid K \in \mathcal{K}_3(\beta)\}$ as $\{\mathcal{G}_n(\beta) \mid n \in \omega\}$. Choose $\mathcal{F}_n(\alpha)$ so that it refines both $\mathcal{F}_n(\beta)$ and $\mathcal{G}_n(\beta)$ and again satisfies (1)–(3).

Suppose α is a limit ordinal and that $\mathcal{F}(\beta)$ has been chosen for all $\beta < \alpha$. Choose the $\mathcal{F}_n(\alpha)$s so that (in addition to (1)–(3)):

(4) For $\beta < \alpha$ and $r \in \omega$ there is $n \in \omega$ such that $\mathcal{F}_n(\alpha)$ refines $\mathcal{F}_r(\beta)$, and

(5) For all $n \in \omega$ there are $\alpha_n < \alpha$ and $n_\alpha \in \omega$ such that $\mathcal{F}_n(\alpha) = \mathcal{F}_{n_\alpha}(\alpha_n)$.

By (5), for limit α and $K \in \mathcal{K}_3(\alpha)$, we can and do choose $F(K)$ and $F'(K)$ from $\bigcup_{\beta<\alpha} \mathcal{F}(\beta)$.

One can prove by a straightforward induction argument that this is possible for all limit $\alpha < \omega_1$ (see [2]).

For $\alpha < \omega_1$, $\mathcal{K}(\alpha)$ partitions X into disjoint compact sets and, if $\beta < \alpha$, every member of $\mathcal{K}(\alpha)$ is a subset of some member of $\mathcal{K}(\beta)$. If $\beta < \alpha$ and $K \in \mathcal{K}_3(\alpha)$, K is a proper subset of some member of $\mathcal{K}_3(\beta)$.

If $\beta \leqslant \alpha$, $K \in \mathcal{K}_3(\beta)$, $K' \in \mathcal{K}_3(\alpha)$ and $K \cap K' = \emptyset$, then there are disjoint F and F' in $\mathcal{F}_m(\beta)$ for some $m \in \omega$ with $K \subset F$ and $K' \subset F'$. Thus the same proof given for Lemma 1 yields:

Lemma 1′. *If $K \neq K'$ in $\bigcup_{\alpha<\omega_1} \mathcal{K}_3(\alpha)$ and $K \cap K' = \emptyset$, there is atmost one $i < 3$ with $V_i(K') \cap K \neq \emptyset$.*

Hence we also have:

Lemma 2′. *There do not exist infinitely many disjoint $K \in \bigcup_{\alpha<\omega_1} \mathcal{K}_3(\alpha)$ with the same $F(K)$ and $F'(K)$.*

We are assuming $\mathcal{K}_3(\alpha) \neq \emptyset$ for any $\alpha < \omega_1$.

For all limit α, choose $K(\alpha) \in \mathcal{K}_3(\alpha)$. Since $F(K(\alpha))$ and $F'(K(\alpha))$ are in $\bigcup_{\beta<\alpha} \mathcal{F}_\beta$, by the pressing down lemma, there are F and F' and an uncountable subset A of the limit ordinals such tht $F(K(\alpha)) = F$ and $F'(K(\alpha)) = F'$ for all $\alpha \in A$.

If $\tau \in \omega_1$, let $\mathcal{A}_\tau = \{A' \subset (A - \tau)\} \mid \{K(\alpha) \mid \alpha \in A'\}$ is a maximal disjoint subset of $\{K(\alpha) \mid \alpha \in (A - \tau)\}$. Each $A' \in \mathcal{A}_\tau$ is finite and we choose $A_\tau \in \mathcal{A}_\tau$ of minimal cardinality.

Since the cardinalities of the A_τs can only increase there is $\sigma < \omega_1$ such that $|A_\sigma| = |A_\tau|$ for all $\tau > \sigma$. Choose an uncountable $T \subset (\omega_1 - \sigma)$ such that $\sigma \in T$ and $\tau' < \tau$ in T implies $\tau > \alpha$ for all $\alpha \in A_{\tau'}$. Choose α minimal in A_σ. For each $\tau \in T$ there is a unique $\alpha_\tau \in A_\tau$ with $K(\alpha_\tau) \subset K(\alpha)$. Since $\{K(\alpha_\tau) \mid \tau \in T\}$ is an uncountable strictly decreasing sequence of compact sets in the separable, monotonically normal space in which closed set must be G_δ [5], we have a contradiction.

2. Constructions of types 1 and 2

Having proved Theorem 1, fix a breakdown $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ of X with associated $\mathcal{K}$ having $\mathcal{K}_3 = \emptyset$. Our primary concern now is $\mathcal{K}_2 = \{K \in \mathcal{K} \mid |K| = 2\}$. For $K = \{K_0, K_1\} \in \mathcal{K}_2$ choose disjoint $W_0(K)$ and $W_1(K)$ with $K_0 \in W_0(K)$ and $K_1 \in W_1(K)$ and, for $i < 2$, define $V_i(K) = H(K_i, W_i(K))$. Define

$$\mathcal{F}(K) = \{F \in \mathcal{F} \mid \overline{F} \subset (V_0(K) \cup V_1(K))\};$$

there is an $n \in \omega$ such that $K \subset F \in \mathcal{F}_n$ implies $F \in \mathcal{F}(K)$. We now have a stronger form of Lemma 1, namely, *if $K \neq K'$ in $\mathcal{K}_2$ and $F \in \mathcal{F}(K) \cap \mathcal{F}(K')$, then K is contained in exactly one of $V_0(K')$ and $V_1(K')$*. Use $i \neq i'$, $j \neq j'$, and $k \neq k'$ for members of 2.

If $F \in \mathcal{F}$ and $B \subset \overline{F}$ is compact, let

$$\mathcal{K}_F(B) = \{K \in \mathcal{K}_2 \mid K \subset B \text{ and } F \in \mathcal{F}(K)\}.$$

If $K \neq K'$ in $\mathcal{K}_F(B)$, there are i and j with $K' \subset V_i(K)$, $K \subset V_j(K')$ and, since H is an MN operator, $V_{i'}(K) \cap V_{j'}(K) = \emptyset$; so $(V_{i'}(K) \cap B) \subset V_j(K')$ and $(V_{j'}(K') \cap B) \subset V_i(K)$.

Define $\langle \mathcal{L}, \leqslant \rangle$ to be a *linked family in $\mathcal{K}_F(B)$* if $\mathcal{L} \subset \mathcal{K}_F(B)$ and $\leqslant$ is a total order on $\mathcal{L}$ such that $K < K' < K''$ in $\langle \mathcal{L}, \leqslant \rangle$, $K' \subset V_i(K)$, and $K' \subset V_j(K'')$, imply

$K'' \subset V_i(K)$ and $K \subset V_j(K'')$. If $K < K'$ in $\langle \mathcal{L}, \leqslant \rangle$ and $K' \subset V_i(K)$ then $K < K''$ in $\mathcal{L}$ if and only if $K'' \subset V_i(K)$, and $K'' < K$ in $\mathcal{L}$ if and only if $K'' \subset V_{i'}(K)$. So $\langle \mathcal{L}, \leqslant \rangle$ induces a total order $\leqslant$ on $\bigcup \mathcal{L}$ with $K_{i'} < K_i$ in this case. Thus, for a given $\mathcal{L} \subset \mathcal{K}_F(B)$ there can be at most two orders which make $\langle \mathcal{L}, \leqslant \rangle$ into a linked family, namely the one which for some $K \in \mathcal{L}$ induces $K_{i'} < K_i$ and the one which induces $K_i < K_{i'}$, namely its' inverse. Without confusion, I hope, I use $\leqslant$ for the order on $\mathcal{L}$ as well as the induced order on $\bigcup \mathcal{L}$.

Suppose the $\langle \mathcal{L}, \leqslant \rangle$ is a maximal linked family in $\mathcal{K}_F(B)$. Reindex each $L \in \mathcal{L}$ so that $L_0 < L_1$ in $\langle \bigcup \mathcal{L}, \leqslant \rangle$.

Lemma 3. *Suppose $K \in (\mathcal{K}_F(B) - \mathcal{L})$, $\mathcal{L}_0 = \{L \in \mathcal{L} \mid K \subset V_1(L)\}$, and $\mathcal{L}_1 = \{L \in \mathcal{L} \mid K \subset V_0(L)\}$. Neither $\mathcal{L}_0$ nor $\mathcal{L}_1$ is empty and there is $i < 2$ such that $\bigcup \mathcal{L} \subset V_i(K)$.*

Proof. Observe that $\mathcal{L}_0$ and $\mathcal{L}_1$ partition $\langle \mathcal{L}, \leqslant \rangle$ into disjoint subintervals with $\mathcal{L}_0 < \mathcal{L}_1$. If $\mathcal{L}_0 = \emptyset$, $\leqslant$ can be extended to $\mathcal{L} \cup \{K\}$ by putting K first and, similarly, if $\mathcal{L}_1 = \emptyset$, by putting K last. By definition, $V_0(L) \cap V_1(L') = \emptyset$ if $L < L'$ in $\langle \mathcal{L}, \leqslant \rangle$, so there are i and j with $\bigcup \mathcal{L}_0 \subset V_i(K)$ and $\bigcup \mathcal{L}_1 \subset V_j(K)$. If $i \neq j$, $\mathcal{L}_0 \neq \emptyset$, and $\mathcal{L}_1 \neq \emptyset$, $\leqslant$ can be extended to $\mathcal{L} \cup \{K\}$ by putting K between $\mathcal{L}_0$ and $\mathcal{L}_1$ with the induced order on $(\bigcup \mathcal{L} \cup \{K\})$ being the one with $K_i < K_j$. All of these extensions contradict the maximality of $\mathcal{L}$. $\square$

Lemma 4. *Suppose $\emptyset \neq \mathcal{L}' \subset \mathcal{L}$ and $A = \bigcup\{V_0(L) \cap B \mid L \in \mathcal{L}'\}$. Then either $\langle \mathcal{L}', \leqslant \rangle$ has a maximal element L, $(\bigcup \mathcal{L}' - L_1) \subset A$, and $A = V_0(L) \cap B$ or there is a unique $a \in (\overline{A} - A)$, $a \in \overline{\bigcup \mathcal{L}'}$, and $\bigcup \mathcal{L}' \subset A$.*

Proof. If $\langle \mathcal{L}', \leqslant \rangle$ has a maximal element L, then if $L' \in \mathcal{L}' - \{L\}$, $(L' \cup (V_0(L') \cap B)) \subset (V_0(L) \cap B)$; so $A = V_0(L) \cap B$. So assume $\langle \mathcal{L}', \leqslant \rangle$ has no maximal element. Again $L' < L$ in $\langle \mathcal{L}', \leqslant \rangle$ implies $(L' \cup (V_0(L') \cap B)) \subset (V_0(L) \cap B)$, so $\bigcup \mathcal{L}' \subset A$. Since $L_0 \notin V_0(L')$ and B is compact, there must be some $a \in \overline{\bigcup \mathcal{L}'} - A$. Suppose there were $a' \neq a$ in $(\overline{A} - A)$. Choose disjoint open U and U' with $a \in U$ and $a' \in U'$. Without loss of generality, there are $L' < L$ in $\mathcal{L}'$ with $V_0(L) \cap H(a', U') \cap B \neq \emptyset$ and $V_0(L') \cap H(a, U) \cap B \neq \emptyset$. Since $a' \in B$ and $a' \notin V_0(L)$, $a' \in V_1(L)$; thus $a' \notin W_0(L)$ and, since $V_0(L) = H(L_0, W_0(L))$, $L_0 \in U'$. Since $B \subset (V_0(L) \cup V_1(L))$, $(B \cap W_0(L)) = V_0(L) \subset A$ and $a \notin W_0(L)$. But $(V_0(L') \cap B) \subset V_0(L) = H(L_0, W_0(L))$. Since $H(a, U) \cap H(L_0, W_0(L)) \neq \emptyset$ and H is an MN operator we have a contradiction. $\square$

Lemma 4′. *Suppose $\emptyset \neq \mathcal{L}' \subset \mathcal{L}$ and $Z = \bigcup\{V_1(L) \cap B \mid L \in \mathcal{L}'\}$. Then either $\langle \mathcal{L}', \leqslant \rangle$ has a minimal element L, $(\bigcup \mathcal{L}' - L_0) \subset Z$, and $Z = V_1(L) \cap B$, or there is a unique $z \in (\overline{Z} - Z)$, $z \in \overline{\bigcup \mathcal{L}'}$, and $\bigcup \mathcal{L}' \subset Z$.*

Construction 1. Given $F \in \mathcal{F}$, a compact $B \subset \overline{F}$, a maximal linked family $\langle \mathcal{L}, \leqslant \rangle$ from $\mathcal{K}_F(B)$ and some $G \in \mathcal{F}$. Index the members of $\mathcal{L}$ so that $L_0 < L_1$ for all $L \in \mathcal{L}$.

For all $x \in B$, let $A_x = \bigcup\{V_0(L) \cap B \mid L \in \mathcal{L}$ and $x \notin V_0(L)\}$ and $Z_x = \bigcup\{V_1(L) \cap B \mid L \in \mathcal{L}$ and $x \notin V_1(L)\}$. Let $M_x = B - (A_x \cup Z_x)$ and $\mathcal{M} = \{M_x \mid x \in B\}$; $\mathcal{M}$

partitions B into disjoint compact sets. Let $\langle\mathcal{M},\leqslant\rangle$ be the total order on $\mathcal{M}$ defined by $M_x \leqslant M_y$ if $A_x \subset A_y$.

Suppose $M = M_x \in \mathcal{M}$. If $\mathcal{L}' = \{L \in \mathcal{L} \mid V_0(L) \subset A_x\} \neq \emptyset$, then, by Lemma 4, either $\mathcal{L}'$ has a maximal element L and $A_x = V_0(L) \cap B$ and $L_1 \in M$ or there is a unique $a_M \in (\overline{A}_x - A)$ with $a_M \in M$. Similarly if $\mathcal{L}'' = \{L \in \mathcal{L} \mid V_1(L) \subset Z_x\} \neq \emptyset$, then, by Lemma 4′, either $\mathcal{L}''$ has a minimal element L and $Z_x = V_1(L) \cap B$ and $L_0 \in M$ or there is a unique $z_M \in (\overline{Z}_x - Z)$ with $z_M \in M$. Let

$$\mathcal{M}^* = \{M \in \mathcal{M} \mid M \not\subset \{a_M, z_M\}\}.$$

By Lemma 3 and the maximality of $\langle\mathcal{L},\leqslant\rangle$, if $K \in (\mathcal{K}_F(B) - \mathcal{L})$, there is an $i < 2$ so that $\bigcup\mathcal{L} \subset V_i(K)$. Reindex K so $\bigcup\mathcal{L} \subset V_0(K)$ (and hence $\bigcup\mathcal{L} \cap V_1(K) = \emptyset$).

Suppose $M \in \mathcal{M}^*$. For all $\alpha < 2^{|X|}$ for which it is possible, choose an open V_α by induction as follows. If possible choose a nonempty maximal linked family $\langle\mathcal{L}_\alpha, \leqslant_\alpha\rangle$ from $\mathcal{K}_F(M) - \bigcup_{\beta<\alpha} V_\beta$ with $K_0 <_\alpha K_1$ for all $K \in \mathcal{L}_\alpha$. If there is a term of $\mathcal{K}_F(M) - \bigcup_{\beta<\alpha} V_\beta$ contained in G, make sure $\mathcal{L}_\alpha$ has such a term. Define

$$V_\alpha = \bigcup\{V_1(K) \mid K \in \mathcal{L}_\alpha\} \quad \text{and} \quad C_\alpha = B \cap V_\alpha.$$

If $\kappa = \{\alpha \mid V_\alpha \text{ is defined}\}$, let $\mathcal{C}_M = \{C_\alpha \mid \alpha < \kappa\}$ and, if $C = C_\alpha$, let $\mathcal{L}_C = \mathcal{L}_\alpha$ and $\leqslant_C = \leqslant_\alpha$.

Suppose $\beta < \alpha < \kappa$ and $L \in \mathcal{L}_\alpha$. By Lemma 3 either $\mathcal{L}_\beta \subset V_0(L)$ or $\mathcal{L}_\beta \subset V_1(L)$. Suppose $\mathcal{L}_\beta \subset V_1(L)$. If there is some $K \in \mathcal{L}_\beta$ such that $L \subset V_1(K)$, then $K \subset V_1(L)$ and $L \subset V_1(K)$ implies $V_0(L) \cap V_0(K) = \emptyset$ contradicting $\mathcal{L} \subset V_0(L) \cap V_0(K)$. So, for all $K \in \mathcal{L}_\beta$, $L \subset V_0(K)$ and $K \subset V_1(L)$. But this contradicts the maximality of $\mathcal{L}_\beta$. So $\mathcal{L}_\beta \subset V_0(L)$ for all $L \in \mathcal{L}_\alpha$.

For each $C \in \mathcal{C}_M$, by Lemma 4, either there is a minimal $K \in \mathcal{L}_C$ in which case $K_0 = q_C \in M - C$ and $C = V_1(K) \cap B$, or there is a unique $q_C \in (\overline{C} - C)$. Defining $C^* = C \cup \{q_C\}$, C^* is a compact subset of M.

Define $D_M = M - \bigcup\mathcal{C}_M$. Let $\mathcal{C} = \bigcup\{\mathcal{C}_M \mid M \in \mathcal{M}^*\}$ and $\mathcal{D} = \{D_M \mid M \in \mathcal{M}^*\}$. Then we call $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{D}$ as just defined, "the type 1 construction for $\langle F, B, \langle\mathcal{L},\leqslant\rangle, G\rangle$".

Lemma 5. *If $\mathcal{C}' \subset \mathcal{C}$ and $x \in \overline{\bigcup\mathcal{C}'} - \bigcup\{C^* \mid C \in \mathcal{C}'\}$, then $x \in \overline{\{q_C \mid C \in \mathcal{C}'\}}$.*

Proof. Without loss of generality $x \neq q_C$ and $x \notin \mathcal{L}_C$ for any $C \in \mathcal{C}'$.

Choose an open U with $x \in U$ and $\overline{U} \cap \{q_C \mid C \in \mathcal{C}'\} = \emptyset$. By the definition of "breakdown", there is an open $U' \subset U$ with $x \in U'$ such that $L \in \mathcal{K}_2$ and $L \not\subset U$ implies $L \cap U' = \emptyset$. This is obvious if $\{x\} \in \mathcal{K}$. If $x = K_i \in K \in \mathcal{K}_2$ let $U^* = (V_i(K) \cap U) \cup V_{i'}(K)$. Again it is obvious that there is an open $U'' \subset U$ with $K \subset U''$ such that $L \in \mathcal{K}_2$ and $L \not\subset U^*$ implies $L \cap U'' = \emptyset$. By an argument analogous to the proof of Lemma 1, if $L \in \mathcal{K}_2$, $L \neq K$, and $L \subset U^*$ either $L \subset V_i(K)$ or $L \subset V_{i'}(K)$. Thus $U' = V_i(K) \cap U \cap U''$ have the desired property.

Choose $C \in \mathcal{C}'$ such that $C \cap H(x, U') \neq \emptyset$. There is some $L \in \mathcal{L}_C$ with $B \cap V_1(L) \cap H(x, U') \neq \emptyset$. For any such L, since $x \in B - V_1(L)$ and $C \supset (V_1(L) \cap B) = (W_1(L) \cap B)$, $x \notin W_1(L)$. Since H is an MN operator $L_1 \in U'$ and, by our choice of U', $L_0 \in U$.

If $\mathcal{L}_C$ has a first term L' we can assume $L' = L$ since $V_1(L) \subset V_1(L')$. In this case $q_C = L_0$ and $C = V_1(L) \cap B$. So $L_0 = q_C \in U$ contradicting our choice of U.

If $q_C \in (\overline{C} - C)$ and there is some $L \in \mathcal{L}_C$ such that $B \cap V_1(L) \cap H(x, U') \neq \emptyset$, then, since $q_C \notin U$, there is some $L' < L$ in $\langle \mathcal{L}_C, \leqslant \rangle$ with $L' \cap U = \emptyset$. Since $(B \cap V_1(L)) \subset (B \cap V_1(L'))$, $B \cap V_1(L') \cap H(x, U') \neq \emptyset$. Thus $L'_0 \notin U$ is a contradiction. □

Construction 2. Given $F \in \mathcal{F}$ and a compact $B \subset \overline{F}$. Choose a maximal linked family $\langle \mathcal{L}, \leqslant \rangle$ from $\mathcal{K}_F(B)$ and index its members so $L_0 < L_1$ for all $L \in \mathcal{L}$ induces the $\leqslant$ order. Also let $\{G_t \mid t \in \omega\}$ be some indexing of $\mathcal{F}$. Then:

Define $\mathcal{C}_0 = \{B\}$, $B^* = B$, and $\langle \mathcal{L}_B, \leqslant_B \rangle = \langle \mathcal{L}, \leqslant \rangle$.

Having chosen $\mathcal{C}_t$ and C^* and $\langle \mathcal{L}_C, \leqslant_C \rangle$ for all $C \in \mathcal{C}_t$, let $\mathcal{M}(C)$, $\mathcal{M}^*(C)$, $\mathcal{C}(C)$, and $\mathcal{D}(C)$ be the type 1 construction for $\langle F, C^*, \langle \mathcal{L}_C, \leqslant_C \rangle, G_t \rangle$. Define

$$\mathcal{D}_t = \bigcup \{\mathcal{D}(C) \mid C \in \mathcal{C}_t\} \quad \text{and} \quad \mathcal{C}_{t+1} = \bigcup \{\mathcal{C}(C) \mid C \in \mathcal{C}_t\}.$$

Recall that in the type 1 construction we indexed all of $\bigcup \mathcal{K}_F(B)$ so that for $K \in (\mathcal{K}_F(B) - \mathcal{L})$, $\bigcup \mathcal{L} \subset V_0(K)$. Observe that we need no further indexing once it is done for $\mathcal{K}_F(B)$ since, by induction, if this order is correct for some $B' \in \mathcal{C}_t$, then for each $C \in \mathcal{C}(B')$ we chose $\langle \mathcal{L}_C, \leqslant_C \rangle$ so $L_0 <_C L_1$ for every $L \in \mathcal{L}_C$ and for all $K \in (\mathcal{K}_F(C^*) - \mathcal{L}_C)$, $V_0(K) \supset \bigcup \mathcal{L}_C$ because $V_0(K) \supset \bigcup \mathcal{L}_{B'}$.

Define $\mathcal{D}^*_\omega = \{\bigcap_{t \in \omega} C_t \mid C^*_t \in \mathcal{C}_t \text{ and } C_{t+1} \in \mathcal{C}(C_t)\}$.

Suppose $D = \bigcap_{t \in \omega} C_t \in \mathcal{D}^*_\omega$ as above. Then for each $t \in \omega$ there is $L_t \in \mathcal{L}_{C_t}$ such that $C_{t+1} \subset V_1(L_t)$. Thus $\{L_t \mid t \in \omega\}$ is a linked family with the induced $(L_t)_0 < (L_t)_1$ order on its union. By Lemma 4, if $A = \bigcup \{V_0(L_t) \cap B \mid t \in \omega\}$ there is a unique $q_D \in D$ in $(\overline{A} - A)$. If $q_D \in K \in \mathcal{K}_F(B)$, then $\bigcup \mathcal{L}_{C_t} \subset V_0(K)$ for all t, so, since $q_D \in \overline{\bigcup_{t \in \omega} \mathcal{L}_{C_t}}$, $q_D = K_0$. If $q_D \in K \in \mathcal{K}_F(B)$ and $K \subset D$, let $\mathcal{D}(D) = \{D \cap V_0(K), D \cap V_1(K)\}$; if $D = \{q_D\}$, let $\mathcal{D}(D) = \emptyset$; otherwise let $\mathcal{D}(D) = \{D\}$.

Define $\mathcal{D}_\omega = \bigcup \{\mathcal{D}(D) \mid D \in \mathcal{D}^*_\omega\}$.

Let $\mathcal{C} = \bigcup_{t \in \omega} \mathcal{C}_t$ and $\mathcal{D} = \bigcup_{t \leqslant \omega} \mathcal{D}_t$ denote the "result of Construction 2 for $\langle B, F \rangle$".

Comment. Since separable monotonically normal spaces are hereditarily separable, the separability of X implies that many things are countable. In Construction 1, $\mathcal{M}^*$ is countable since $\mathcal{M}^*$ includes precisely those members of $\mathcal{M}$, a collection of disjoint sets, containing a point not in the closure of the union of the others. Also, for each $M \in \mathcal{M}^*$, $\mathcal{C}_M$ is a set of disjoint, relatively open subsets of M, so $\mathcal{C}_M$ is countable. Thus $\mathcal{C}$ is countable. Hence when we get to Construction 2, by induction, each $\mathcal{C}_t$ is countable and $\mathcal{C}$ is countable. In Construction 1, $\mathcal{D}$ consists of precisely one closed subset of each of the compact, disjoint $M \in \mathcal{M}^*$. So in Construction 2, by induction, each $\mathcal{D}_t$ and $\bigcup_{t \in \omega} \mathcal{D}_t$ consist of countable families of disjoint compact sets. The members of $\mathcal{D}^*_\omega$ are disjoint, compact, and disjoint from $\bigcup_{t \in \omega} \mathcal{D}_t$. Although $\mathcal{D}^*_\omega$ may not be countable, $\mathcal{D}(D) = \emptyset$ for $D \in \mathcal{D}^*_\omega$ unless D contains a point not in the closure of $\bigcup(\mathcal{D}^*_\omega - \{D\})$ and each $\mathcal{D}(D)$ which is not empty partitions D into at most two compact subsets. Thus $\mathcal{D}$ is a countable family of disjoint compact sets.

Lemma 6. *If $D \in \mathcal{D}$, $\mathcal{K}_F(D) = \emptyset$.*

Proof. Suppose $D \in \mathcal{D}_t$ for some finite t. There is $C \in \mathcal{C}_t$ and $M \in \mathcal{M}^*(C)$ with $D = D_M$. For every $\alpha < 2^{|X|}$ for which it was possible we chose $\mathcal{L}_\alpha \subset (\mathcal{K}_F(M) - \bigcup_{\beta<\alpha} V_\beta)$ and $V_\alpha \subset (X - D)$. Thus there is no term of $\mathcal{K}_F(M)$ contained in D; but $D \subset M$ so $\mathcal{K}_F(D) = \emptyset$.

Suppose $D \in \mathcal{D}(E)$ for some $E \in \mathcal{D}^*_\omega$ and $K \in \mathcal{K}_F(D)$. By our choice of $\mathcal{D}(E)$, $q_E \notin K$. So $K \subset (E - \{q_E\})$ which is open in B. There is $t \in \omega$ with $K \subset (G_t \cap B) \subset (E - \{q_E\})$. There is a unique $C \in \mathcal{C}_t$ with $E \subset C$, and $\mathcal{L}_C$ was chosen to have a member contained in G_t if *any* member of $\mathcal{K}_F(C)$ is in G_t. But $\mathcal{K}_F(D) \subset \mathcal{K}_F(E) \subset \mathcal{K}_F(C)$ so $\exists L \in \mathcal{L}_C$ with $L \subset G_t$ and $L \subset (E - \{q_E\})$. However, no $L \in \mathcal{L}_C$ is contained in any $M \in \mathcal{M}(C)$, and $E \subset M$ for some $M \in \mathcal{M}(C)$ so we have a contradiction. □

3. Main theorem

Theorem 2. *There is a separable, 0-dimensional, compact, monotonically normal space Δ and continuous map π from Δ onto X.*

3.1. Construction of π and Δ

Suppose $n \in \omega$. For $F_n \in \mathcal{F}_n$, let

$$F_n^* = \{x \in X \mid \forall m < n\ \exists F_m \in \mathcal{F}_m \text{ with } \overline{F_{m+1}} \subset F_m\}.$$

Observe that $\{F^* \mid F \in \mathcal{F}_n\}$ is a finite closed cover of X. Let $\Sigma_n = \{\langle D_0, F_0, D_1, F_1, \ldots, D_n, F_n\rangle \mid D_0 = X,\ \forall m\ F_m \in \mathcal{F}_m$, and $\forall m < n$, $\overline{F_{m+1}} \subset F_m$ and $D_{m+1} \in \mathcal{D}$ of the type 2 construction for $\langle D_m \cap F_m^*, F_m\rangle\}$.

Let $\Sigma_\omega = \{\sigma = \langle D_0, F_0, D_1, F_1, \ldots\rangle \mid \forall n \in \omega$, σ extends some term of $\sigma_n\}$. Let $\Sigma = \bigcup_{n\leqslant\omega} \Sigma_n$. If $\sigma \in \Sigma_\omega$, let $\sigma^* = \bigcap \sigma$. If $x \in \sigma^*$, $x \in \bigcap_{n\in\omega} F_n = K \in \mathcal{K}$. If $K = \{x\}$, $\sigma^* = \{x\}$. If $x = K_i$, there is $n \in \omega$ with $F_n \in \mathcal{F}(K)$ and thus, by Lemma 6, $K \not\subset D_{n+1}$, so $\sigma^* = \{x\}$ even in this case. Let $\Delta_\omega = \{\langle x, \sigma\rangle \mid \sigma \in \Sigma_\omega$ and $\sigma^* = \{x\}\}$.

Suppose $n \in \omega$ and $\sigma = \langle D_0, F_0, \ldots, D_n, F_n\rangle \in \Sigma_n$; let $\sigma^* = D_n \cap F_n^*$. Let $\mathcal{C}(\sigma)$ and $\mathcal{D}(\sigma)$ be the type 2 construction for $\langle \sigma^*, F_n\rangle$. Then say:

(1) If $C \in \mathcal{C}(\sigma)$, $M \in \mathcal{M}(C)$, and a_M exists,

$$\langle a_M, \sigma, C, 0\rangle \in \Delta_\sigma.$$

(2) If $C \in \mathcal{C}(\sigma)$, $M \in \mathcal{M}(C)$ and z_M exists,

$$\langle z_M, \sigma, C, 1\rangle \in \Delta_\sigma.$$

(3) If $D \in \mathcal{D}^*_\omega(\sigma)$, $\langle q_D, \sigma\rangle \in \Delta_\sigma$.

These three cases constitute Δ_σ and $\Delta_n = \bigcup\{\Delta_\sigma \mid \sigma \in \Sigma_n\}$, while $\Delta = \bigcup\{\Delta_n \mid n \leqslant \omega\}$.

Define $\pi : \Delta \to X$ by $\pi(\langle x, \sigma\rangle) = x$ and $\pi(\langle x, \sigma, C, k\rangle) = x$.

To define the topology on Δ we need further definitions. Assume $n \in \omega$ and $\sigma \in \Sigma_n$.

If $C \in \mathcal{C}(\sigma)$ and q_C exists choose $\gamma = \Gamma(C,\sigma) \in \Sigma_m$ extending σ with $q_C \in \gamma^*$ so that $m \leqslant \omega$ is maximal. Since $q_C \in D \in \mathcal{D}(\sigma)$, γ properly extends σ and, since m is maximal, $q_C \notin \bigcup \mathcal{D}(\gamma)$.

Suppose $A \subset \sigma^*$. Define $\Delta(A,\sigma) = \{\delta \in \Delta \mid \delta \text{ is } \langle x,\tau\rangle \text{ or } \langle x,\tau,C,K\rangle, \tau \text{ extends } \sigma, \text{ and } x \in A\}$. If $C \in \mathcal{C}(\sigma)$ and $q_C \in \overline{C}$, let $\Delta^*(C,\sigma) = \Delta(C,\sigma) \cup \{\langle q_C,\sigma,C,1\rangle\}$; note that $q_C \notin C$ but $q_C = z_M$ where $M = \{q_C\}$ is the first term of $\langle \mathcal{M}(C), \leqslant\rangle$ in this case. Otherwise let $\Delta^*(C,\sigma) = \Delta(C,\sigma)$. Define $\mathcal{C}_0(A,\sigma) = \{\langle C,\tau\rangle \mid q_C \in A,\ C \in \mathcal{C}(\tau), \text{ and } \Gamma(C,\tau) \text{ extends } \sigma \text{ which properly extends } \tau\}$. If $\mathcal{A} \subset \mathcal{C}_0(A,\sigma)$, define $\mathcal{C}_0(A,\sigma,\mathcal{A}) = (\mathcal{C}_0(A,\sigma) - \mathcal{A})$. For $m < n-1$, define $\mathcal{C}_{m+1}(A,\sigma,\mathcal{A}) = \{\langle B,\rho\rangle \in \mathcal{C}_0(C,\tau) \mid \langle C,\tau\rangle \in \mathcal{C}_m(A,\sigma,\mathcal{A})\}$. Define $\mathcal{C}(A,\sigma,\mathcal{A}) = \bigcup_{m<n} \mathcal{C}_m(A,\sigma,\mathcal{A})$. Finally $\Delta(A,\sigma,\mathcal{A}) = \Delta(A,\sigma) \cup \bigcup\{\Delta^*(C,\tau) \mid \langle C,\tau\rangle \in \mathcal{C}(A,\sigma,\mathcal{A})\}$. Observe that $\{\Delta^*(C,\tau) \mid \langle C,\tau\rangle \in \mathcal{C}(A,\sigma,\mathcal{A})\}$ are disjoint and their union misses $\Delta(A,\sigma)$; also σ properly extends τ if $\langle C,\tau\rangle \in \mathcal{C}(A,\sigma,\emptyset)$, $\Delta(A,\sigma,\mathcal{A}) \subset \Delta(A,\sigma,\emptyset)$, and $\Delta(A,\sigma,\emptyset) \cap \Delta(A',\sigma,\emptyset) = \emptyset$ unless $A \cap A' \neq \emptyset$.

Similarly, suppose $\delta = \langle x,\sigma,B,k\rangle \in \Delta_\sigma$. Define $\mathcal{C}_0(\delta) = \{\langle C,\tau\rangle \in \mathcal{C}_0(\{x\},\sigma) \mid \Gamma(C,\tau) = \sigma \text{ and, if } x = a_M \text{ for some } M \in \mathcal{M}(B), \text{ then } k = 0\}$. If $\mathcal{A} \subset \mathcal{C}_0(\delta)$ then define $\mathcal{C}_M(\delta,\mathcal{A})$, $\mathcal{C}(\delta,\mathcal{A})$, and $\Delta(\delta,\mathcal{A})$ exactly as above, replacing $(A,\sigma,\mathcal{A})$ by $(\delta,\mathcal{A})$.

Let $\mathcal{U}(\sigma)$ be the set of all sets of the following types:

(1) $\Delta(\sigma^*,\sigma,\mathcal{A})$ where $\mathcal{A}$ is a finite subset of $\mathcal{C}_0(\sigma^*,\sigma)$.

(2) $\Delta(C,\sigma,\mathcal{A}) \cup \{\langle q_C,\sigma,C,1\rangle\}$ where $C \in \mathcal{C}(\sigma)$ and $\mathcal{A}$ is a finite subset of $\mathcal{C}_0(\overline{C},\sigma)$.

(3) $\Delta(C-D,\sigma,\emptyset) \cup \{\langle q_C,\sigma,C,1\rangle\} \cup \{\langle q_D,\sigma\rangle\}$ where $D \in \mathcal{D}^*_\omega(\sigma)$ and $D \subset C \in \mathcal{C}(\sigma)$.

(4) $\Delta(\delta,\mathcal{A}) \cup \Delta(\bigcup\{N \mid M < N < M' \text{ in } \langle \mathcal{M}(C),\leqslant\rangle\},\sigma,\emptyset) \cup \Delta(\beta,\mathcal{A}')$ where $C \in \mathcal{C}(\sigma)$, $M < M'$ in $\langle \mathcal{M}(C),\leqslant\rangle$, $\mathcal{A}$ is a finite subset of $\mathcal{C}_0(\delta)$, $\mathcal{A}'$ is a finite subset of $\mathcal{C}_0(\beta)$, $\delta = \langle z_M,\sigma,C,1\rangle$ and $\beta = \langle a_{M'},\sigma,C,0\rangle$.

Just delete any undefined terms from the above such as the $\{\langle q_C,\sigma,C,1\rangle\}$ in (2) or (3) in case $q_C \neq z_M$ for any $M \in \mathcal{M}(C)$, or $\Delta(\delta,\mathcal{A})$ or $\Delta(\beta,\mathcal{A}')$ in case there is no z_M or $a_{M'}$ in (4).

Let

$$\mathcal{U} = \bigcup\{\mathcal{U}(\sigma) \mid \sigma \in \Sigma_n,\ n \in \omega\}.$$

To check that $\mathcal{U}$ *is the basis for a topology on* Δ we suppose that $\delta \in U \cap V$ for some $U \in \mathcal{U}(\gamma)$ and $V \in \mathcal{U}(\rho)$ and check that there is a member of $\mathcal{U}$ to which δ belongs contained in $U \cap V$.

If $\delta \in \Delta_\sigma$ for some finite σ we can assume $\gamma = \rho = \sigma$. Then, if $\delta = \langle x,\sigma\rangle$, $x = q_D$ for some $D \in \mathcal{D}^*_\omega(\sigma)$ and we can assume there are C and C' in $\mathcal{C}(\sigma)$ with U being the type (3) member of $\mathcal{U}(\sigma)$ for C and D and V the one for C' and D. One of C and C' contains the other; say $C \subset C'$. Then $\delta \in U \subset V$. A similar argument using type (4) basic neighborhoods for δ where $\delta = \langle x,\sigma,C,k\rangle$ for some $C \in \mathcal{C}(\sigma)$ and $k < 2$ takes care of δ for all finite σ.

If $\delta = \langle x,\tau\rangle \in \Delta_\omega$, there is a finite σ extended by τ and properly extending both ρ and γ. Then, for some finite $\mathcal{A} \subset \mathcal{C}_0(\sigma^*,\sigma)$, $\Delta(\sigma^*,\sigma,\mathcal{A}) \subset U$, and, for some finite $\mathcal{A}' \subset \mathcal{C}_0(\sigma^*,\sigma)$, $\Delta(\sigma^*,\sigma,\mathcal{A}') \subset V$; so $\delta \in \Delta(\sigma^*,\sigma,\mathcal{A} \cup \mathcal{A}') \subset U \cap V$.

3.2. Proof that π and Δ have the desired properties for our theorem

Point are closed. To see this suppose $\delta \neq \beta$ in δ and δ is $\langle x,\tau\rangle$ or $\langle x,\tau,B,k\rangle$. We get a contradiction by assuming $\delta \in \mathcal{U}$ implies $\beta \in U$.

If $\delta \in \Delta_\omega$ there is a finite σ extended by τ such that either τ is not extended by σ or σ has greater length than τ. Then $\delta \in \Delta(\sigma^*,\sigma,\emptyset) \in \mathcal{U}$. If $\beta \in \Delta(\sigma^*,\sigma,\emptyset)$ there is some unique $\langle C,\gamma\rangle \in \mathcal{C}_0(\sigma^*,\sigma,\emptyset)$ such that $\Gamma(C,\gamma)$ extends σ and $\beta \in \Delta^*(C,\gamma) \cup \Delta(C,\gamma,\emptyset)$. So if $\mathcal{A} = \{\langle C,\gamma\rangle\}$, $\delta \in U = \Delta(\sigma^*,\sigma,\mathcal{A})$ while $\beta \notin U$.

But assuming τ is finite, the intersection of the members of $\mathcal{U}(\tau)$ for δ is the singleton $\{\delta\}$; so at least one member of $\mathcal{U}(\tau)$ avoids β.

Δ is zero dimensional. This time assume $U \in \mathcal{U}(\sigma)$ for some finite $\sigma \in \Sigma$, $\delta \in \Delta_\tau$ is $\langle x,\tau\rangle$ or $\langle x,\tau,C,k\rangle$ for some $\tau \in \Sigma$, and $\delta \notin U$. We get a contradiction by assuming $\delta \in V \in \mathcal{U}$ implies $U \cap V \neq \emptyset$.

Many of the four types of members of $\mathcal{U}(\sigma)$ involve an $\mathcal{A}$ (or two). If $\delta \in \Delta^*(C,\gamma) \cup \Delta\mathcal{C}(C,\gamma,\emptyset)$ for some $\langle C,\gamma\rangle \in \mathcal{A}$, $\Delta^*(C,\gamma) \cup \Delta(C,\gamma,\emptyset)$ is an open set containing δ and missing U. So we can assume $\mathcal{A} = \emptyset$. If $\delta \in (\Delta(\sigma^*,\sigma,\emptyset) - U)$ and U is of type (2), (3) or (4), there is an open set to which δ belongs missing U, so we can assume $U = \Delta(\sigma^*,\sigma,\emptyset)$.

If $\delta = \langle x,\tau\rangle \in \Delta_\omega$ there is a finite ρ extended by τ of length greater than that of σ; clearly $\delta \in \Delta(\rho^*,\rho,\emptyset)$. Let

$$\mathcal{A} = \{\langle A,\alpha\rangle \in \mathcal{C}_0(\rho^*,\rho) \mid \exists\langle A',\alpha'\rangle \in (\{\langle A,\alpha\rangle\} \cup \mathcal{C}(A,\alpha,\emptyset)) \text{ such that } \Delta(\sigma^*,\sigma) \subset \Delta^*(A',\alpha')\}.$$

Since $\{\Delta^*(A',\alpha') \mid \langle A',\alpha'\rangle \in \mathcal{C}(\rho^*,\rho,\emptyset)\}$ are disjoint, $\mathcal{A}$ has at most one term and $\delta \in \Delta(\rho^*,\rho,\mathcal{A}) = V$. Assume there is $\beta \in U \cap V$ since otherwise V will do for our desired open set. Then $\beta \in \Delta^*(C,\gamma) \cap \Delta^*(C,\gamma')$ for some $\langle C,\gamma\rangle \in \mathcal{C}(\sigma^*,\sigma,\emptyset)$ and $\langle C',\gamma'\rangle \in \mathcal{C}(\rho^*,\rho,\mathcal{A})$.

Because of β, either γ extends γ' or γ' extends γ and if $\gamma = \gamma'$ either $C \subset C'$ or $C' \subset C$.

If γ' properly extends γ or $\gamma' = \gamma$ and $C' \subsetneq C$, there is $D \in \mathcal{D}(\gamma)$ with $\rho^* \subset D \subset C$, so ρ extends γ and $\rho^* \subset C$, hence $\delta \in \Delta(\rho^*,\rho) \subset \Delta^*(C,\gamma) \subset U$ contradicting $\delta \notin U$. Similarly, if γ properly extends γ' or $\gamma' = \gamma$ and $C \subsetneq C'$, $\Delta(\sigma^*,\sigma) \subset \Delta^*(C',\gamma')$ contradicting our choice of $\mathcal{A}$.

So $\gamma = \gamma'$ and $C = C'$; also $C \neq \rho^*$ since this implies $\delta \in U$. Therefore $\langle C,\gamma\rangle \in \mathcal{C}(\rho^*,\rho,\mathcal{A}) \cap \mathcal{C}(\sigma^*,\sigma,\emptyset)$. Let us assume that γ has maximal length for there to be some $\langle \mathcal{C},\gamma\rangle$ in this intersection. There is $\langle A,\alpha\rangle \in \{\langle\sigma^*,\sigma\rangle\} \cup \mathcal{C}(\sigma^*,\sigma,\emptyset)$ and $\langle A',\alpha'\rangle \in \{\langle\rho^*,\rho\rangle\} \cup \mathcal{C}(\rho^*,\rho,\mathcal{A})$ such that $q_C \in A \cap A'$ and $\Gamma(q_C,\gamma)$ extends both α and α', either α extends α' or α' extends α. Because of $q_C \in A \cap A'$, if $\alpha = \alpha'$ either $A \subset A'$ or $A' \subset A$. So the argument from the preceding paragraph implies $\alpha = \alpha'$ and $A = A'$ and $\langle A,\alpha\rangle \in \mathcal{C}(\rho^*,\rho,\mathcal{A}) \cap \mathcal{C}(\sigma^*,\sigma,\emptyset)$ contradicting α has length greater than γ.

It remains to check the situation if δ is $\langle x,\tau,C,k\rangle$ or $\langle x,\tau\rangle$ for some finite τ. Since $\delta \notin U$, τ does not extend σ. So if σ does not properly extend τ, the same argument given for $\delta \in \Delta_\omega$ applies, replacing ρ by τ. Otherwise σ properly extends τ and there is $D \in \mathcal{D}(\tau)$ with $\sigma^* \subset D$ and thus $U = \Delta(\sigma^*, \sigma, \emptyset) \subset \Delta(D, \tau, \emptyset)$. There is a basic neighborhood for δ in $\mathcal{U}(\tau)$, of type (4) if $\delta = \langle x, \tau, C, k\rangle$ and type (3) if $\delta = \langle x, \tau\rangle$, missing $\Delta(D, \tau)$; this neighborhood misses U.

Δ is separable. Recall that X is hereditarily separable and that $\pi^{-1}(x)$ is countable for every $x \in X$. For each $n \in \omega$ and $\sigma \in \Sigma_n$, σ^* has a countable dense in the topology of X subset S_σ. Thus $\pi^{-1}(S_\sigma)$ is countable and its closure contains Δ_σ. Since $\bigcup_{n\in\omega} \Sigma_n$ is countable, $S_\Delta = \bigcup\{\pi^{-1}(S_\sigma) \mid \sigma \in \Sigma_n \text{ and } n \in \omega\}$ is countable. If $\langle x, \sigma\rangle \in \Delta_\omega$ then, for each $n \in \omega$, there is a unique $\sigma_n \in \Sigma_n$ extended by σ, and $x \in \sigma_n^*$. Since every open set to which $\langle x, \sigma\rangle$ belongs contains $\Delta(\sigma_n^*, \sigma_n, \mathcal{A})$ for some finite $\mathcal{A} \subset \mathcal{C}_0(\sigma_n^*, \sigma_n)$, and $\Delta(\sigma_n^*, \sigma_n, \mathcal{A}) \cap \pi^{-1}(S_{\sigma_n}) \neq \emptyset$. So $\Delta \subset \overline{S}_\Delta$.

Δ is compact. Assume $\mathcal{V} \subset \mathcal{U}$ is a cover of Δ and define $\mathcal{V}^* = \{\Omega \subset \Delta \mid \text{some finite subset of } \mathcal{V} \text{ covers } \Omega\}$. We prove $\Delta \in \mathcal{V}^*$ by assuming $\Delta \notin \mathcal{V}^*$. Thus, since Σ_0 is finite and $X = \bigcup\{\Delta(\sigma^*, \sigma) \mid \sigma \in \Sigma_0\}$, there is $\sigma_0 \in \Sigma_0$ such that $\Delta(\sigma_0^*, \sigma_0) \notin \mathcal{V}^*$.

If, for all $n \in \omega$, there is $\sigma_n \in \Sigma_n$ extending σ_m for all $m < n$ and $\Delta(\sigma_n^*, \sigma_n) \notin \mathcal{V}^*$, then, if $\sigma \in \Sigma_\omega$ extends σ_n for all $n \in \omega$, $\langle \sigma^*, \sigma\rangle \in V \in \mathcal{V}$ and there is $n \in \omega$ with $\Delta(\sigma_n^*, \sigma_n, \mathcal{A}) \subset V$ contradicting $\Delta(\sigma_n^*, \sigma_n) \notin \mathcal{V}^*$. Hence there is $\sigma \in \Sigma_n$ for some finite n such that $\Delta(\sigma^*, \sigma) \notin \mathcal{V}^*$ but $\Delta(\tau^*, \tau) \in \mathcal{V}^*$ for all $\tau \in \Sigma_{n+1}$ extending σ.

Recall that to achieve $\mathcal{C}(\sigma)$ and $\mathcal{D}(\sigma)$ for $\sigma = \langle D_0, F_0, \ldots, D_n, F_n\rangle$ we did a type 2 construction for $\langle \sigma^*, F_n\rangle$ and for each $t \in \omega$ we defined a family $\mathcal{C}_t(\sigma)$ of subset of σ^* and for each $C \in \mathcal{C}_t(\sigma)$ we did a type 1 construction for $\langle F_n, C^*, \langle \mathcal{L}_C, \leqslant\rangle, G_t\rangle$ yielding $\mathcal{M}(C)$, $\mathcal{C}(D)$ and $\mathcal{D}(C)$.

Since $\mathcal{C}_0(\sigma) = \{\sigma^*\}$, $C_0 \in \mathcal{C}_0(\sigma)$ implies $\Delta(C_0, \sigma) \notin \mathcal{V}^*$. If, for all $t \in \omega$, there is $C_t \in \mathcal{C}_t(\sigma)$ with $C_{t+1} \subset C_t$ and $\Delta^*(C_t, \sigma) \notin \mathcal{V}^*$, there is $D = \bigcap_{t\in\omega} C_t \in \mathcal{D}_\omega^*(\sigma)$. Since $\langle q_D, \sigma\rangle \in V \in \mathcal{V}$, there are a basic open set of type (3) for $\langle q_D, \sigma\rangle$ in V and a $t \in \omega$ with $\Delta(C_t - D, \sigma) \subset V$. Since $\Delta^*(C_t, \sigma) \notin \mathcal{V}^*$, $\Delta(D, \sigma) \notin \mathcal{V}^*$ and one, call it D_{n+1}, of the at most two members of $\mathcal{D}_\omega(\sigma)$ whose union is D, has $\Delta(D_{n+1}, \sigma) \notin \mathcal{V}^*$. Since $\mathcal{F}_{n+1}$ is finite there is some $F_{n+1} \in \mathcal{F}_{n+1}$ such that $\Delta(D_{n+1} \cap F_{n+1}^* \sigma) \notin \mathcal{V}^*$. But if $\tau = \langle D_0, \ldots, F_n, D_{n+1}, F_{n+1}\rangle$, there is at most $\langle q_D, \sigma\rangle$ in $\Delta(D_{n+1} \cap F_{n+1}^*, \sigma) - \Delta(\tau^*, \tau)$ so $\Delta\langle \tau^*, \tau\rangle \notin \mathcal{V}^*$ which is a contradiction. Hence there is $C \in \mathcal{C}(\sigma)$ with $\Delta^*(C, \sigma) \notin \mathcal{V}^*$ such that $B \in \mathcal{C}(\sigma)$ and $B \subset C$ implies $\Delta^*(B, \sigma) \in \mathcal{V}^*$.

Since $\langle \mathcal{M}(C), \leqslant\rangle$ is Dedekind complete with a first and last term and the topology of Δ respects the order topology on $\langle \mathcal{M}(C), \leqslant\rangle$ (see the definition of type (4) members of $\mathcal{U}(\sigma)$), there must be some $M \in \mathcal{M}(C)$ such that $\Delta(M, \sigma) \notin \mathcal{V}^*$. Since $\mathcal{F}_{n+1}$ is finite $T = \{\langle D_0, \ldots, F_n D_M, F\rangle \mid F \in \mathcal{F}_{n+1}\}$ is finite, and, since $\tau \in T$ implies $\tau \in \Sigma_{n+1}$ and extends σ, there is an open V in Δ with $\bigcup\{\Delta(\tau^*, \tau) \mid \tau \in T\} \subset V \in \mathcal{V}$. Recall that $M = D_M \cup (\bigcup \mathcal{C}(M))$ and each $B \in \mathcal{C}(M)$ has $\Delta^*(B, \sigma) \in \mathcal{V}^*$. Let $\mathcal{B} = \{B \in \mathcal{C}_M \mid \Delta^*(B, \sigma) \not\subset V\}$. Since

$$\Delta(M, \sigma) \subset V \cup \bigcup \bigl(\{\Delta^*(B, \sigma) \mid B \in \mathcal{B}\} \cup \{\langle q_M, \Sigma, C, 0\rangle\} \cup \{\langle z_M, \sigma, C, 1\rangle\}\bigr),$$

$\mathcal{B}$ is infinite.

Choose $\tau_{n+1} \in T$ extended by $\Gamma(q_B, \sigma)$ for infinitely many $B \in \mathcal{B}$ and, for all $m > n+1$ for which it is possible, choose $\tau_m \in \Sigma_m$ extending τ_{m-1} and extended by $\Gamma(q_B, \sigma)$ for infinitely many $B \in \mathcal{B}$. If $\tau \in \Sigma_\omega$ extends τ_m for all m, $\langle \tau^*, \tau \rangle \in \Delta(\tau^*_{n+1}, \tau_{n+1}) \subset V$. So for some m and $\mathcal{A}$, $\Delta(\tau^*_m, \tau_m \mathcal{A}) \subset V$. If $B \in \mathcal{B}' = \{B \in \mathcal{B} \mid \Gamma(q_B, \sigma) \text{ extends } \tau_m\}$, $\langle B, \sigma \rangle \in \mathcal{C}_0(\tau^*_m, \tau_m)$ and, since $\mathcal{B}'$ is infinite, there is $\langle B, \sigma \rangle \in (\mathcal{B}' - \mathcal{A})$ and $\Delta^*(B, \sigma) \subset V$ contrary to assumption. Thus there is a maximal m for which τ_M can be defined, call it τ. Then $\mathcal{B}^+ = \{B \in \mathcal{B} \mid \Gamma(q_B, \sigma) = \tau\}$ is infinite and, if $B \in \mathcal{B}^+$, $q_B \notin \bigcup \mathcal{D}(\tau)$. Again $\mathcal{C}_0(\tau) = \{\tau^*\}$. If for all $t \in \omega$ there is $C_t \in \mathcal{C}_t(\tau)$ with $C_{t+1} \subset C_t$ such that $\{B \in \mathcal{B}^+ \mid q_B \in C_t\}$ is infinite, there is $D = \bigcap_{t \in \omega} C_t \in \mathcal{D}^*_\omega(\tau)$, and since $\langle q_D, \tau \rangle \in V$, for some t, $\Delta(C_t - D, \tau, \emptyset) \subset V$. Since $D \cap \{q_B \mid B \in \mathcal{B}^+\} = \emptyset$, there is $B \in \mathcal{B}^+$ with $q_B \in C_t$. Thus $\langle B, \sigma \rangle \in \mathcal{C}_0(C_t, \tau)$; so $\Delta^*(B, \sigma) \subset \Delta(C_t - D, \tau, \emptyset) \subset V$ which is impossible. Hence there is $C' \in \mathcal{C}(\tau)$ such that $\{B \in \mathcal{B}^+ \mid q_B \in C'\}$ is infinite but, for all $B' \in \mathcal{C}(\tau)$ with $B' \subsetneq C'$, $\{B \in \mathcal{B}^+ \mid q_B \in B'\}$ is finite.

Again the basic open sets for $\langle a_{M'}, \tau, 0 \rangle$ and $\langle a_{M'}, \tau, 1 \rangle$ for $M' \in \langle \mathcal{M}(C'), \leqslant \rangle$ ensure that there is some $M' \in \mathcal{M}(C')$ such that $\{B \in \mathcal{B}^+ \mid q_B \in M'\}$ is infinite. Since $\{B \in \mathcal{B}^+ \mid q_B \in D_{M'}\} = \emptyset$ and $M' = D_{M'} \cup (\bigcup \mathcal{C}(M'))$ and $\{B \in \mathcal{B}^+ \mid q_B \in B'\}$ is finite for $B' \in \mathcal{C}(M')$, $\mathcal{B}_0 = \{B' \in \mathcal{C}(M') \mid \{B \in \mathcal{B}^+ \mid q_B \in B'\} \neq \emptyset\}$ is infinite. Observe that $\langle B, \sigma \rangle \in \mathcal{C}_0(B', \tau)$ if $B \in \mathcal{B}^+$ and $q_B \in B' \in \mathcal{C}(M')$.

To sum up our situation so far: We have a finite $\sigma \in \Sigma$, a $C \in \mathcal{C}(\sigma)$, an $M \in \mathcal{M}(C)$, and an open $V \in \mathcal{V}^*$ such that $\mathcal{B} = \{B \in \mathcal{C}(M) \mid \Delta^*(B, \sigma) \not\subset V\}$ is infinite. We also have a finite $\tau \in \Sigma$ properly extending σ with $\Delta(\tau^*, \tau) \subset V$, a $C' \in \mathcal{C}(\tau)$, and an $M' \in \mathcal{M}(C')$ such that $\mathcal{B}_0 = \{B' \in \mathcal{C}(M') \mid \exists B \in \mathcal{B} \text{ with } \langle B, \sigma \rangle \in \mathcal{C}_0(B', \tau)\}$ is infinite.

Let $\gamma_0 = \tau$ and, using exactly the same steps used in the construction of τ, C', M', and $\mathcal{B}_0$ from σ, C, M, and $\mathcal{B}$, for each $r \in \omega$ we can choose a finite $\gamma_r \in \Sigma$ properly extending γ_{r-1}, a $C_r \in \mathcal{C}(\gamma_r)$, and an $M_r \in \mathcal{M}(C_r)$ such that $\mathcal{B}_r = \{B' \in \mathcal{C}(M_r) \mid \exists B \in \mathcal{B}_{r-1} \text{ with } \langle B, \sigma \rangle \in \mathcal{C}_0(B', \tau_r)\}$ is infinite.

There is $\gamma \in \Sigma_\omega$ extending γ_r for all $r \in \omega$ and, since $\langle \gamma^*, \gamma \rangle \in V$, there are $r \in \omega$ and $\mathcal{A} \subset \mathcal{C}_0(\gamma^*_r, \gamma_r)$ such that $\Delta(\gamma^*_r, \gamma_r, \mathcal{A}) \subset V$. If $r = 0$, $\gamma_0 = \tau$ and $\{\langle B, \sigma \rangle \mid B \in \mathcal{B}\} \subset \mathcal{C}_0(\tau^*, \tau)$ and there is $B \in \mathcal{B}$ with $\langle B, \sigma \rangle \notin \mathcal{A}$. So $\Delta^*(B, \sigma) \subset V$ which is a contradiction. If $r > 0$, $\{\langle B', \gamma_{r-1} \rangle \mid B' \in \mathcal{B}_{r-1}\} \subset \mathcal{C}_0(\gamma^*_r, \gamma_r)$, and there is $B' \in \mathcal{B}_{r-1}$ with $\langle B', \gamma_{r-1} \rangle \notin \mathcal{A}$. Since $\Delta(B', \gamma_{r-1}, \emptyset) \subset \Delta(\gamma^*_r, \gamma_r, \mathcal{A})$, $\Delta(B', \gamma_{r-1}, \emptyset) \subset B$. But there is some $B \in \mathcal{B}$ with $\langle B, \sigma \rangle \in \mathcal{C}(B', \gamma_{r-1}, \emptyset)$ which implies $\langle B, \sigma \rangle \subset V$ with is impossible.

π is continuous. Assume $x \in U$ which is open in X. We prove that $\delta \in \pi^{-1}(x)$ implies $\delta \in V$ for some open V in Δ with $\pi(V) \subset U$.

If $\delta = \langle x, \rho \rangle \in \Delta_\omega$ there is $\sigma \in \Sigma_n$ for some finite n with ρ extending σ and $x \in \sigma^* = A \subset U$. If δ is $\langle x, \sigma, C, k \rangle \in \Delta_n$ for some $k < 2$, there is $M \in \mathcal{M}(C)$ so that $x = a_M$ if $k = 0$ and $x = z_M$ if $k = 1$. If $k = 0$ choose $M' < M$ in $\langle \mathcal{M}(C), \leqslant \rangle$ so that $A = (\{x\} \cup \bigcup \{N \in \mathcal{M}(C) \mid M' \leqslant N < M\}) \subset U$ and if $k = 1$ choose $M'' > M$ in $\langle \mathcal{M}(C), \leqslant \rangle$ so that $A = (\{x\} \cup \bigcup \{N \in \mathcal{M}(C) \mid M < N \leqslant M''\}) \subset U$. If

$\delta = \langle x, \sigma\rangle \in \Delta_n$ choose $C \in \mathcal{C}(\sigma)$ and $D \in \mathcal{D}^*_\omega(\sigma)$ so that $x = q_D \in A = \overline{C - D} \subset U$. In all cases A is a compact subset of $\sigma^* \cap U$ and trivially $\pi(\Delta(A, \sigma)) \subset U$. Observe that if $\mathcal{A}$ is a finite subset of $\mathcal{C}_0(A, \sigma)$ and $\overline{B} \subset U$ for all $\langle B, \gamma\rangle \in \mathcal{C}(A, \sigma, \mathcal{A})$, then there is a basic open set $V \in \mathcal{U}(\sigma)$ with $\delta \in V$ and, since each $\pi(\Delta(B, \gamma)) \subset U$, $\pi(V) \subset \pi(\Delta(A, \sigma, \mathcal{A})) \subset U$.

For all $m \leqslant n$ choose an open U_m in X with $A \subset U_0 \subset \overline{U}_0 \subset U_1 \subset \overline{U}_1 \subset \cdots \subset \overline{U}_n \subset U$. If γ is properly extended by σ let $\mathcal{B}_{0\gamma} = \{B \in \mathcal{C}(\gamma) \mid (B, \gamma) \in \mathcal{C}_0(A, \sigma)\}$ and $\mathcal{A}_{0\gamma} = \{B \in \mathcal{B}_{0\gamma} \mid B \not\subset U_0\}$. We claim $\mathcal{A}_{0\gamma}$ is finite. If $\sigma = \langle D_0, F_0, \ldots, D_n, F_n\rangle$ and $\gamma = \langle D_0, F_0, \ldots, D_m, F_m\rangle$, then $\sigma^* \subset D_{m+1} \in \mathcal{D}(\gamma)$. If $B \in \mathcal{B}_{0\gamma}$, $q_B \in \sigma^* \subset D_{m+1} = D_M$ for some $M \in \mathcal{M}(\gamma)$ and $B \in \mathcal{C}(M)$. Thus $\{B - U_0 \mid B \in \mathcal{A}_\gamma\}$ is a set of disjoint, nonempty, compact subsets of $X - U_0$ whose union, if $\mathcal{A}_{0\gamma}$ is infinite, has, by Lemma 5, a limit point in σ^*, a compact subset of U_0. This is clearly impossible, so $\mathcal{A}_{0\gamma}$ is finite. Note that $A_{0\langle\gamma\rangle} = \bigcup(\mathcal{B}_{0\gamma} - \mathcal{A}_{0\gamma}) \subset U_0 \cap \gamma^*$ and $\overline{A_{0\langle\gamma\rangle}} \subset \overline{U}_0 \cap \gamma^* \subset U_1 \cap \gamma^*$.

Let $\Omega = \{\langle \gamma_0, \gamma_1, \ldots, \gamma_m\rangle \mid m \leqslant n$, γ_0 properly extends σ, and γ_r properly extends γ_{r-1} if $r > 0\}$. By induction on m, for each $E = \langle \gamma_0, \ldots, \gamma_m\rangle \in \Omega$ we choose $A_{mE} \subset U_m \cap \gamma^*_m$ (as well as $\mathcal{B}_{mE}$ and $\mathcal{A}_{mE}$). Having chosen $A_{m-1E'} \subset U_{m-1} \cap \gamma^*_{m-1}$ for $E' = \langle \gamma_0, \ldots, \gamma_{m-1}\rangle$, we define $\mathcal{B}_{mE} = \{B \in \mathcal{C}(\gamma_m) \mid (B, \gamma_m) \in \mathcal{C}_0(\overline{A_{(m-1)E'}}, \gamma_{m-1})\}$, $\mathcal{A}_{mE} = \{B \in \mathcal{B}_{mE} \mid B \not\subset U_m\}$, and $A_{mE} = \bigcup(\mathcal{B}_{0\gamma} - \mathcal{A}_{0\gamma}) \subset U_m \cap \gamma^*_m$. Repeating the argument for the construction of $A_{0\langle\gamma\rangle}$, we learn that $\mathcal{A}_{mE}$ is finite for all m and E.

If $E = \langle \gamma_0, \ldots, \gamma_m\rangle \in \Omega$ and $r < m$, let $E(r) = \langle \gamma_0, \ldots, \gamma_r\rangle$. If $B \in \mathcal{A}_{mE}$ define $B_m = B$ and for $r < m$, define B_r to be the unique term of $\mathcal{B}_{rE(r)} - \mathcal{A}_{rE(r)}$ such that $q_{B_{(r+1)}} \in B_r$. Let $\mathcal{A} = \{B_0 \mid B \in \mathcal{B}_{mE}$ for some $m \leqslant n$ and $E \in \Omega$ having m terms$\}$. Since Ω is finite, $\mathcal{A}$ is finite. Since for all $\langle B, \gamma\rangle \in \mathcal{C}(A, \sigma, \mathcal{A})$, $B \subset U$ (and $A \subset U$), $\pi(\Delta(A, \sigma, \mathcal{A})) \subset U$ as desired.

Δ is monotonically normal. It suffices to define a monotone normality operator G for Δ.

Suppose $\delta \in V$ which is open in Δ. If $\delta \in \Delta_\omega$ there are $n \in \omega$, $\sigma \in \Sigma_n$ and a finite $\mathcal{A} \subset \mathcal{C}_0(\sigma^*, \sigma)$ with $\delta \in \Delta(\sigma^*, \sigma, \mathcal{A}) \subset V$. Choose n (which determines σ) minimal, and having made this choice choose $\mathcal{A}$ with $|\mathcal{A}|$ minimal. Then $\mathcal{A}$ is also uniquely determined since $\delta \in \Delta(\sigma^*, \sigma, \mathcal{A}) \cap \Delta(\sigma^*, \sigma, \mathcal{A}')$ implies $\delta \in \Delta(\sigma^*, \sigma, \mathcal{A} \cap \mathcal{A}')$. Define $G(\delta, V) = \Delta(\sigma^*, \sigma, \mathcal{A})$.

For all finite $\sigma \in \Sigma$ and $C \in \mathcal{D}(\sigma)$, let $\langle \mathcal{M}(C), \prec\rangle$ be a well ordering of $\mathcal{M}(C)$.

If $\delta = \langle x, \sigma, C, 0\rangle \in \Delta - \Delta_\omega$, there is $M \in \mathcal{M}(C)$ such that $x = a_M$. Since $\langle \mathcal{M}(C), \leqslant\rangle$ is Dedekind complete there is a minimal $M' < M$ in $\langle \mathcal{M}(C), \leqslant\rangle$ such that $[\Delta(\bigcup\{N \mid M' < N < M\}, \sigma, \emptyset) \cup \Delta(\delta, \mathcal{A})] \subset V$ for some $\mathcal{A}$. Choose $\mathcal{A}$ minimal for M'. Since $\preccurlyeq$ is a well, ordering there is minimal M^* for $M' < M^* < M$ in $\langle \mathcal{M}(C), \preccurlyeq\rangle$. Define $G(\delta, V) = \Delta(\bigcup\{N \mid M^* < N < M\}, \sigma, \emptyset) \cup \Delta(\delta, \mathcal{A})$.

Similarly, if $\delta = \langle x, \sigma, C, 1\rangle \in \Delta - \Delta_\omega$, there is $M \in \mathcal{M}(C)$ such that $x = z_M$. Choose $M' > M$ maximal in $\langle \mathcal{M}(C), \leqslant\rangle$ for $[\Delta(\bigcup\{N \mid M < N < M'\}, \sigma, \emptyset) \cup (\Delta, \mathcal{A})] \subset V$. Choose $\mathcal{A}$ minimal. There is a minimal M^* for $M < M^* < M'$ in $\langle \mathcal{M}(C), \preccurlyeq\rangle$. Define $G(\delta, V) = \Delta(\bigcup\{N \mid M < N < M^*\}, \sigma, \emptyset) \cup \Delta(\delta, \mathcal{A})$.

If $\delta = \langle x, \sigma\rangle \in \Delta - \Delta_\omega$, $x = q_D$ for some $D \in \mathcal{D}^*_\omega(\sigma)$ and there is $C \in \mathcal{C}(\sigma)$ such that $\Delta(C - D, \sigma, \emptyset) \subset V$. Choose C maximal for this to be so. Since the members of $\mathcal{C}(\sigma)$ containing D are reverse well ordered by inclusion this is possible. Define $G(\delta, V) = \Delta(C - D, \sigma, \emptyset) \cup \{\delta\}$.

In all of the above cases we say $G(\delta, V) \in \mathcal{G}(\sigma)$. If E is a closed subset of Δ and $E \subset V$, open, define $G(E, V) = \bigcup\{G(\delta, V) \mid \delta \in E\}$. Clearly G is monotone, each $G(E, V)$ is open, and $E \subset G(E, V) \subset V$. Thus to prove G is a monotonic normality operator for Δ it suffices to assume that $\delta \in V$, open, and $\beta \in W$, open, $\delta \notin W$ and $\beta \notin V$, and prove that $G(\delta, V) \cap G(\beta, W) = \emptyset$. Suppose $G(\delta, V) \cap G(\beta, W) \neq \emptyset$.

Suppose that $G(\delta, V) \in \mathcal{G}(\sigma)$. Then $\delta \in \Delta_\omega$ implies $G(\delta, V) = \Delta(A_\delta, \sigma, \mathcal{A}_\delta)$ where $A_\delta = \sigma^*$ and $\mathcal{A}_\delta \subset \mathcal{C}_0(\sigma^*, \sigma)$. If $\delta = \langle x, \sigma \rangle \in \Delta_\sigma$, $G(\delta, V) = \Delta(A_\delta, \sigma, \emptyset) \cup \{\delta\}$ and $A_\delta \subset \sigma^*$. If $\delta = \langle x, \sigma, C, k \rangle$, $G(\delta, V) = \Delta(A_\delta, \sigma, \emptyset) \cup \Delta(\delta, \mathcal{A}_\delta)$ where $\mathcal{A}_\delta \subset \mathcal{C}_0(\delta)$ and $A_\delta \subset \sigma^*$. Suppose $G(\beta, W) \in \mathcal{G}(\tau)$ and define A_β and $\mathcal{A}_\beta$ analogously. Let us now prove that if $\sigma = \tau$, $A_\beta \cap A_\delta = \emptyset$.

Assume $\sigma = \tau$. Clearly neither β nor δ is in Δ_ω. So assume δ is $\langle x, \sigma, E, k \rangle$ or $\langle x, \sigma \rangle$ and β is $\langle y, \sigma, E', h \rangle$ or $\langle y, \sigma \rangle$. There is a maximal $t \in \omega$ such that for some $C_t \in \mathcal{C}_t(\sigma)$ both x and y are in $\overline{C}_t$. (Since $\{x, y\} \subset (\bigcap_{t \in \omega} C_t) = D_\omega \in \mathcal{D}^*_\omega(\sigma)$ implies $\delta = \beta = \langle q_D, \sigma \rangle$.) So $x \in M_x \in \mathcal{M}(C_t)$ and $y \in M_y \in \mathcal{M}(C_t)$. If $x \in \overline{C}_x$ for some $C_x \in \mathcal{C}_{t+1}(\sigma)$, with $A_\delta \cap C_x \neq \emptyset$, then $A_\delta \subset C_x$. Otherwise δ is $\langle a_{M_x}, \sigma, C_t, k \rangle$ for some $k < 2$. Thus, if $A_\beta \cap A_\delta \neq \emptyset$, $\delta = \langle a_{M_x}, \sigma, C_t, k \rangle$ and $\beta = \langle z_{M_y}, \sigma, C_t, h \rangle$ for some $k < 2$ and $h < 2$. If $M_x = M_y$, x is a_{M_x} and y is z_{M_x} (or vice versa) and then A_δ is contained in those Ms below M_x in $\langle \mathcal{M}(C_t), \leqslant \rangle$ and A_β in those above. If $M_x \neq M_y$ our choice of M^* ensures the Ms containing A_β and A_δ are disjoint. So $A_\beta \cap A_\delta = 0$ if $\tau = \sigma$.

Now suppose $\sigma \in \Sigma_n$ and $\tau \in \Sigma_r$. Define $P_{\delta 0} = \langle A_\delta, \sigma \rangle$ and for $m < n$, define $P_{\delta(m+1)} = \mathcal{C}_m(A_\delta, \sigma, \mathcal{A}_\delta)$ if $\delta \in \Delta_\omega$, $\mathcal{C}_m(A_\delta, \sigma, \emptyset)$ if $\delta = \langle x, \sigma \rangle$, and $\mathcal{C}_m(A_\delta, \sigma, \emptyset) \cup \mathcal{C}_m(\delta, \mathcal{A}_\delta)$ if $\delta = \langle x, \sigma, E, k \rangle$. Then

$$G(\delta, V) = \{\delta\} \cup \bigcup \left\{ \Delta^*(C, \gamma) \mid (C, \gamma) \in \bigcup_{m \leqslant n} P_{\delta m} \right\}.$$

Define $P_{\beta s}$ for $s \leqslant r$ similarly.

Since $G(\delta, V) \cap G(\beta, W) \neq \emptyset$, there are $m \leqslant n$ and $s \leqslant r$ and $\alpha \in \Delta^*(C, \gamma) \cap \Delta^*(C', \gamma')$ for some $(C, \gamma) \in P_{\delta m}$ and $(C', \gamma') \in P_{\beta s}$ whether α is of the form $\langle p, \eta \rangle$ or $\langle p, \eta, J, j \rangle$, η extends both γ and γ'. If $\gamma \neq \gamma'$, $p \in C \cap C'$ and there is $D \in \mathcal{D}(\gamma)$ with $\tau^* \subset \gamma'^* \subset D$. If $D \subset C$, $\beta \in \Delta(\tau^*, \tau) \subset \Delta(C, \gamma) \subset G(\delta, V)$ contrary to assumption. Otherwise $D \cap C = \emptyset$ contradicting $C \cap C' = \emptyset$.

Thus $\gamma = \gamma'$. We cannot have both $m = 0$ and $r = 0$ since then $\langle C, \gamma \rangle = \langle A_\delta, \sigma \rangle$ and $\langle C', \gamma' \rangle = \langle A_\beta, \sigma \rangle$ and we have shown that $A_\delta \cap A_\beta = \emptyset$ and thus $\Delta(A_\delta, \sigma) \cap \Delta(A_\beta, \tau) = \emptyset$ contrary to assumption. Hence say $r > 0$ so $C' \in \mathcal{C}(\gamma)$. If $m = 0$ and $\gamma = \sigma$, $C = A_\delta$ which is not cut by any $C' \in \mathcal{C}(\sigma)$. If $m > 0$, $C \in \mathcal{C}(\gamma)$ which also has this property. So in any case, either $C' \subset C$ or $C \subset C'$. If $C' \subset C$ but $C \neq C'$, then τ properly extends γ and there is $D \subset C$ such that $D \in \mathcal{D}(\sigma)$ and $\tau^* \subset D$. Thus $\beta \in \Delta(C, \gamma) \subset W(\delta, V)$ contrary to assumption. The same argument shows $C \not\subset C'$ unless $C = C'$; hence $C = C'$.

Since $C = C' \in \mathcal{C}(\gamma)$ and $m = 0$ implies $\gamma = \sigma$ and $C = A_\delta \notin \mathcal{C}(\sigma)$ unless $C = \sigma^*$, and $C' = \sigma^*$ and $\gamma = \sigma$ implies $\delta \in \Delta(\sigma^*, \sigma) \subset \Delta^*(C', \sigma) \subset G(\beta, W)$, $m \neq 0$.

Thus $\langle C, \gamma \rangle = \langle C', \gamma' \rangle \in P_{m\delta} \cap P_{r\beta}$ where both m and r are greater than zero. Thus there are $\langle E, \rho \rangle \in P_{(m-1)\delta}$ and $\langle E', \rho' \rangle \in P_{(r-1)\beta}$ where $\Gamma(C, \gamma)$ extends both ρ and ρ' and $Q_C \in E \cap E'$. These facts are precisely what is needed to apply the same argument again to show that $\langle E', \rho' \rangle = \langle E', \rho' \rangle$ where both $(m-1)$ and $(r-1)$ are greater than zero. Repeating the argument $\leqslant m$ times clearly leads us to a contradiction, thus proving that Δ is monotonically normal.

Since Δ is compact, separable, zero dimensional, and monotonically normal and $\pi : \Delta \to X$ is continuous and onto our theorem is proved. □

References

[1] J. Nikiel, Some problems on continuous images of compact ordered spaces, Questions Answers Gen. Topology 4 (1986) 117–128.

[2] M.E. Rudin, Compact, separable, linearly ordered spaces, Topology Appl. 82 (1998) 397–419.

[3] R.W. Heath, D.J. Lutzer and P.L. Zener, Monotonically normal spaces, Trans. Amer. Math. Soc. 178 (1973) 481–493.

[4] F.P. Ramsey, On a problem of formal logic, Proc. London Math. Soc. 30 (1930) 264–286.

[5] A.J. Ostaszewski, Monotone normality and G_δ-diagonals in the class of inductively generated spaces, in: A. Császár, ed., Topology, Colloquia Mathematica Societatis János Bolyai 23 (North-Holland, Amsterdam, 1980) 905–930.